Groverian Entanglement Measure and Evolution of Entanglement in Search Algorithm for n(=3 And 5)-Qubit Systems with Real Coefficients.

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Evolution of entanglement with the processing of quantum algorithms affects the outcome of the algorithm. Particularly, the performance of Grover's search algorithm gets worsened if the initial state of the algorithm is an entangled one. The success probability of search can be seen as an operational measure of entanglement. This paper demonstrates an entanglement measure based on the performance of Grover's search algorithm for three and five qubit systems. We also show that although the overall pattern shows growth of entanglement, its rise to a maximum and then consequent decay, the presence of local fluctuation within each iterative step is likely.

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Introduction

Quantum entanglement [1] and superposition [2] are the pillars of quantum computation and quantum information theory [3,4]. Quantum information theory has reached the new arenas exploiting these two. Quantum entanglement, inherently a non-classical phenomenon, signifies correlations between quantum systems even if they are space-like separated. In recent times it has been reckoned as a physical resource and hence utilized for various computational tasks including quantum information processing [5] and cryptography [6]. Applications like quantum teleportation can only be materialized if certain amount of entanglement exists between the communicators initially. For this reason, quantification of entanglement of quantum states attains utmost importance. Various entanglement evaluating measures have been figured by current researchers. Methodology based on operational considerations have been successfully employed to formulate entanglement measures for bipartite systems [7,8]. Based on correspondence between thermodynamics and entanglement, entropy of entanglement has been considered as the unique measure of entanglement of pure states [9]. In addition to this, certain attributes have been framed to measure entanglement [10-13]. These attributes are based on axiomatic considerations. According to these, any entanglement should not prevail in product states, it should not vary with local unitary operations and should not increase consequential to any sequence of local operations complemented by only classical communication between parties. Measures satisfying the above properties are called entanglement monotones [11].

Recently an entanglement measure has been developed by Biham et al [14]. The measure is based on the linkage of the success of Grover's search algorithm [15,16] to the amount of entanglement present in the initial state. Performance of Grover's algorithm deteriorates with increasing entanglement in the initial state. Considering the modified quantum search as given in [14] in which a product of arbitrary local operations is applied to intial input register, the formulation of maximal probability of success, $P_{max}(\psi)$, as an entanglement monotone can be precisely made. For a search space containing $N = 2^n$ elements, where n is an integer, the elements can be represented by an n-qubit register and the intial register as $|\phi\rangle$. For a single marked solution, s, to the search problem, P_{max} in terms of the operator U_G^m, representing m Grover iterations may be written as

$$P_{\max} = \max_{U_1,......U_n} \frac{1}{N} \sum_{s=0}^{N-1} \left| \left\langle s \left| U_G^m \left(U_1 \otimes U_2 \otimes ......\otimes U_n \right) \right| \phi \right\rangle \right|^2 \tag{1}$$

by averaging uniformly over all N possible values for s. The maximization is over all local unitary operations $U_1,....U_n$ on the respective qubits of input register state $|\phi\rangle$.

This can be generalized by considering the action of the Grover iterations on the equal superposition state $|\eta\rangle = \sum_x \frac{|x\rangle}{\sqrt{N}}$. Applying in Grover iterations yields

$$U_G^m |\eta\rangle = |s\rangle + O\left(\frac{1}{\sqrt{N}}\right) \tag{2}$$

where the second term is a small correction because Grover's algorithm yields a solution with probability $1 - O\left(\frac{1}{\sqrt{N}}\right)$. Multiplying eq. (2) by $\left(U_G^m\right)^\dagger$ and then taking the Hermitian Conjugate gives

$$\langle s | U_G^m = \langle \eta | + O\left(\frac{1}{\sqrt{N}}\right) \tag{3}$$

Substituting in eq. (1) gives, for a general state $|\phi\rangle$,

$$P_{\max} = \max_{U_1......U_n} \frac{1}{N} \sum_{s=0}^{N-1} \left| \left\langle \eta \left| U_1 \otimes U_2 \otimes .....\otimes U_n \right| \phi \right\rangle \right|^2 + O\left(\frac{1}{\sqrt{N}}\right) \tag{4}$$

Since $|\eta\rangle$ is a product state, eq(4) may equivalently may be expressed as

$$P_{\max} = \max_{|e_1.....,e_n\rangle} \left| \left\langle e_1,....e_n | \phi \right\rangle \right|^2 + O\left(\frac{1}{\sqrt{N}}\right) \tag{5}$$

where the maximization now runs over all product states, $|e_1,.....e_n\rangle = |e_1\rangle \otimes ....|e_n\rangle$, of the n qubits. This suggests that P_{max} depends on the maximum of the overlap between all product states and the input state $|\phi\rangle$. For a product state as input state, P_{max} would be equal to one, whereas with an entangled state as input state, P_{max} would never be one. Success probability of the search algorithm depends on the entanglement of initial register state. Quantifying entanglement following the above approach is related to the performance of the quantum state as an input to the modified search algorithm. The measure thusly referred to as Groverian entanglement can be defined for a state $|\psi\rangle$ by

$$G(\psi) = \sqrt{1 - P_{\max}} \tag{6}$$

. $P_{max}(\psi)$ is an entanglement monotone and consequently $G(\psi)$ too. Following the same line of reasoning authors [17] have examined the success rate of Grover's search algorithm for various four qubit states and Groverian entanglement measure has been worked out for certain kind of input states.

In this letter, the authors have evaluated the success rate of Grover's search algorithm for three and five qubit states and Groverian entanglement measure has been formulated for the same.

Three-qubit states

An arbitrary initial state $|\Psi\rangle$ of three qubits can be written as

$$|\Psi\rangle = \sum_{i=0}^{7} a_i \, |i\rangle \qquad (7)$$

where $|i\rangle = |i_0, i_1, i_2\rangle$.

It can have upto eight terms. Now a general product state of three qubits can be written as

$$|e\rangle = |e_1\rangle \otimes |e_2\rangle \otimes |e_3\rangle \qquad (8)$$

where a single qubit can be represented as

$$|e_k\rangle = \cos\theta_k |0\rangle_k + e^{i\Phi_k} \sin\theta_k |1\rangle_k \qquad (9)$$

where k = 1,2,3. Angle θ_k is in the range $0 \leq \theta_k \leq \pi/2$, while Φ_k is in the range $0 \leq \Phi_k \leq 2\pi$.

The Groverian entanglement measure is derived through the maximization of the function,

$$P(\theta_1, \theta_2, \theta_3, \Phi_1, \Phi_2, \Phi_3, \Psi) = |\langle e|\Psi\rangle|^2 \qquad (10)$$

with respect to variables θ_k, Φ_k (where k = 1, 2, 3) and Ψ.

The product state $|e\rangle$ has supposedly real amplitudes only with all $\Phi_k = 0$ or π if initial state is the one for which all a_i's are real. In order to discuss the success probability of this particular case, the maximization over Φ_k is reduced to a discrete maximization with $e^{i\Phi_k} = \pm 1$. Doubling the range of θ_k to $\pi/2 \leq \theta_k \leq \pi/2$ makes $\sin\theta_k$ to be both positive and negative for same value of $\cos\theta_k$ thereby neutralizing the presence of $i\Phi_k$. Hence, Ψ with real a_i's will have its expression for maximum success probability as

$$P_{max}(\Psi) = \max_{\theta_1,\theta_2,\theta_3} P(\theta_1, \theta_2, \theta_3, \Psi) \qquad (11)$$

The function P is given by

$$P(\theta_1, \theta_2, \theta_3, \Psi) = [a_{000}\cos\theta_1\cos\theta_2\cos\theta_3 + a_{001}\cos\theta_1\cos\theta_2\sin\theta_3 + a_{010}\cos\theta_1\sin\theta_2\cos\theta_3 + a_{011}\cos\theta_1\sin\theta_2 \sin\theta_3 + a_{100}\sin\theta_1 \cos\theta_2\cos\theta_3 + a_{101} \sin\theta_1\cos_2 \sin\theta_3 + a_{110}\sin\theta_1\sin\theta_2\cos\theta_3 + a_{111}\sin\theta_1\sin\theta_2\sin\theta_3]^2 \qquad (12)$$

With the help of trigonometric identities, expression for P can written as,

$$P(\theta_w, \theta_x, \theta_y, \theta_Z, \Psi) = [\frac{a_{000} - a_{110} - a_{101} - a_{011}}{4}\cos\theta_w + \frac{a_{100} + a_{010} + a_{001} - a_{111}}{4}\sin\theta_w + \frac{a_{000} - a_{110} + a_{101} + a_{011}}{4}\cos\theta_x + \frac{a_{100} + a_{010} - a_{001} + a_{111}}{4}\sin\theta_x + \frac{a_{000} + a_{110} - a_{101} + a_{011}}{4}\cos\theta_y + \frac{a_{100} - a_{010} + a_{001} + a_{111}}{4}\sin\theta_y + \frac{a_{000} + a_{110} + a_{101} - a_{011}}{4}\cos\theta_z + \frac{a_{100} - a_{010} - a_{001} - a_{111}}{4}\sin\theta_z]^2 \qquad (13)$$

where $\theta_w = \theta_1 + \theta_2 + \theta_3$, $\theta_x = \theta_1 + \theta_2 - \theta_3$, $\theta_y = \theta_1 - \theta_2 + \theta_3$, $\theta_z = \theta_1 - \theta_2 - \theta_3$.

Maximization of P is obtained by maximizing $P(\theta_w, \theta_x, \theta_y, \theta_Z, \Psi)$ with respect to θ_w, θ_x, θ_y, and $\theta_{Z.}$ $P_{max}(\Psi)$ is thus obtained by satisfying the condition of maxima for $P(\theta_w, \theta_x, \theta_y, \theta_Z, \Psi)$,

$$\text{i.e.,} \ \frac{\partial P}{\partial \theta_w} = \frac{\partial P}{\partial \theta_x} = \frac{\partial P}{\partial \theta_y} = \frac{\partial P}{\partial \theta_z} = 0 \qquad (14)$$

This leads to the following expression for maximum success probability for three qubits case.

$$P_{max} = \frac{1}{16}[\sqrt{(a_{000} - a_{110} - a_{101} - a_{011})^2 + (a_{100} + a_{010} + a_{001} - a_{111})^2} + \sqrt{(a_{000} - a_{110} + a_{101} + a_{011})^2 + (a_{100} + a_{010} - a_{001} + a_{111})^2} + \sqrt{(a_{000} + a_{110} - a_{101} + a_{011})^2 + (a_{100} - a_{010} + a_{001} + a_{111})^2} + \sqrt{(a_{000} + a_{110} + a_{101} - a_{011})^2 + (a_{100} - a_{010} - a_{001} - a_{111})^2}] \qquad (15)$$

With the analytical expression for three qubits, $P_{max}(\Psi)$ can be calculated for various choices of Ψ. For a product state, the entanglement is zero, this can be easily verified. For example uniform product state can be obtained with three qubits in either of these states:

$$\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle).$$

The substitution of a_i's in the analytical expression gives $P_{max}(\Psi) = 1$, and thereby giving $G(\Psi) = 0$.

Eq. (15) makes it possible to evaluate the value of Groverian entanglement measure for a general three qubit state with real coefficients. Some such cases will be discussed.

As discussed earlier (Biham et al, [14]), the search algorithm starts with a product state with zero entanglement, it evolves as the iterative operation proceeds, reaching a maximum, and then decays. It would be of interest to describe the evolution of entanglement for three qubit system. If we consider the state of uniform superpositon which is a linear combination of eight terms

$$|\Psi\rangle = \frac{1}{2\sqrt{2}}\left(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle + |111\rangle \right)$$

and find out the amount of entanglement present in each intermediate state then it is observed that entanglement grows, reaches its maximum and eventually fades away as we reach the desired state. Fig 1. indicates evolution of entanglement within the intermediate states. The intermediate states can be obtained by applying the operators P_W and P_S successively to the initial state. P_W is an operator of the form $1 - 2\,|w\rangle\langle w|$, where $|w\rangle$ is the desired state. P_S is the operator of the form $2\,|\Psi\rangle\langle\Psi| - 1$.

The Groverian entanglement measure $G(\Psi)$ as a function of the no. of iterations has been plotted in Fig. 1. It is of interest to note the change of $G(\Psi)$ within each iterative step: the first step (operator P_W) raises $G(\Psi)$ to around 0.37 and second step (operator P_S) reduces it to 0.26. A similar variation is seen in the second iterarive step where $G(\Psi)$ rises to 0.28 and then finally drops to 0.14. Expectedly the entanglement should approach zero with P_{max} approaching 1. However, it is not usually so, as each iterative operation rotates the state vector in the direction of the desired state with a certain angle θ , depending upon number of qubits, n, which at times rotates the initial state exactly to the desired state [18].

Five-Qubit States

Following the same procedure as earlier, an analytical expression for maximum success probability of Grover's search algorithm for five qubit states has been obtained. The expression is given in the Appendix. A general five qubits state is a linear combination of thirty two terms. Groverian entanglement measure for the same can be calculated. For example, the general product state $|\Psi\rangle$ with uniform superposition is of the form

$$|\Psi\rangle = \sum_{i=0}^{31} a_i\,|i\rangle, \text{ where } |i\rangle = |i_0 \ldots. i_4\rangle \text{ and } a_i = \frac{1}{4\sqrt{2}} \text{ for all i.}$$

On substituting the value of coefficients of each state it can be seen that $P_{max}(\Psi) = 1$, and thus $G(\Psi) = 0$, verifying that a product state has zero entanglement.

Thus maximum success probability and hence Groverian entanglement can be obtained for certain five qubit states from the analytical expression given in the appendix.

Once again it can be shown for a five qubit state that success probability of Grover's search algorithm is affected by the entanglement present in the initial state. Entanglement limits the success probability. Also it can be verified that entanglement grows and then dies out as the iterative operation of the search algorithm goes forward. Fig. 2 displays Groverian entanglement as a function of the no. of iterations of search algorithm. The pattern of variation within each iterative step is non-monotonic and similar to one displayed in fig.1.

von Neumann entropy as an entanglement measure

For a system composed of subsystems A and B, the von Neumann entropy of the reduce density matrix of sub system A is given by

$$S(\rho_A) = -Tr\left[\rho_A \log \rho_A\right] \tag{16}$$

The entanglement of a quantum system can be quantified as bipartite entanglement by calculating its entropy of entanglement, which is expressed as the von Neumann entropy of the reduced state of one of its subsystem. However, for a quantum system with n > 2 subsystems, this quantification cannot be done precisely. In [19], a quantum system with any arbitrary number of subsystems has been considered as bipartite, with one subsystem consisting of a single qubit and the second subsystem all the rest. The reduced density matrix can be calculated for any single qubit because von Neumann entropy is independent of the choice of remaining qubits. The reduced density matrix for the *l*th qubit can be written in its standard form as,

$$\rho_l(k) = \frac{1}{2}\left[I + \vec{s}(k).\vec{\sigma}\right], \tag{17}$$

where components of the Bloch vector $\vec{s}(k)$, for the intermediate states of Grover's search algorithm can be calculated. The state after k Grover iterations becomes

$$|\psi_k\rangle = \frac{\cos\theta_k}{\sqrt{N-1}}\sum_{x\neq y}|x\rangle + \sin\theta_k|y\rangle \tag{18}$$

where $\theta_k = (2k+1)\,\theta_0$ and θ_0 satisfies sin $\theta_0 = 1/\sqrt{N}$, and $N = 2^n$ for an n qubit system. Hence after k iterations, the components of Bloch vector $\vec{s}(k)$ are

$$\begin{aligned} s_x(k) &= \frac{N-2}{N-1}\cos^2\theta_k + \frac{1}{\sqrt{N-1}}\sin 2\theta_k \\ s_y(k) &= 0 \\ s_z(k) &= \frac{1}{N-1}\cos^2\theta_k - \sin^2\theta_k \end{aligned} \tag{19}$$

the von Neumann entropy can thus be calculated as

$$S\left[\rho_l(k)\right] = -tr\left[\rho_l(k)\log\rho_l(k)\right] \tag{20}$$

Following the same approach as in [19], entropy of entanglement has been calculated for intermediate states of Grover's search algorithm for three and five qubit systems. The results have been shown in table 1. It is quite conspicuous from figures 3 and 5 that degree of entanglement as calculated from Groverian entanglement measure and entropy of entanglement as well, for intermediate states of search algorithm follow the same pattern. The presence of local maxima after every P_W rotation can be observed by both the measures. Thus both these measures support the fact that intermediate states of search algorithm through which the system evolves are kentangled. Despite of initial and target states being the product states. Due to lack of any precise measure of entanglement for quantum systems with n > 2 subsystems, it is difficult at the moment to say which of the two measures is more appropriate for measuring actual amount of entanglement present.

Table 1.
Evolution of entanglement within Grover's search algorithm for three and five qubit case.

Starting state	Three qubit case Entropy of Entanglement	Five qubit case Entropy of Entanglement

$\vert\Psi\rangle$(uniform superposition)	0.08	0.14
$P_W \vert\Psi\rangle$	0.84	0.39
$PsP_W\vert\Psi\rangle$(state after 1st iteration)	0.31	0.31
$P_W\ Ps\ P_W \vert\Psi\rangle$	0.54	0.49
$PsP_WPsP_W\vert\Psi\rangle$ (state after 2nd iteration)	0.19	0.47
$P_W\ PsP_WPsP_W\vert\Psi\rangle$		0.49
$PsP_WPsP_WPsP_W\vert\Psi\rangle$(state after 3rd iteration		0.25
$P_W\ PsP_WPsP_WPsP_W\vert\Psi\rangle$		0.31
$PsP_W\ PsP_WPsP_WPsP_W\vert\Psi\rangle$(state after 4th iteration)		0

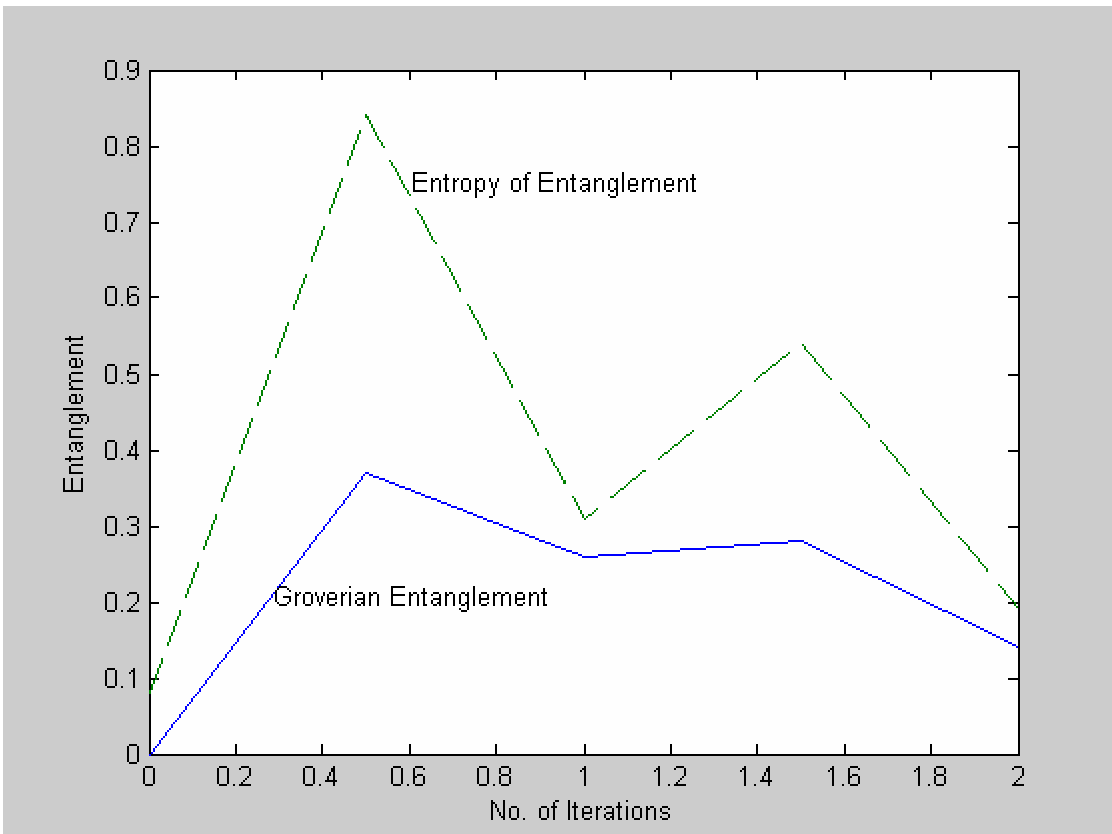

Figure1. Groverian entanglement measure G(Ψ) as a function of the no. of iterations for three qubit quantum search algorithm.

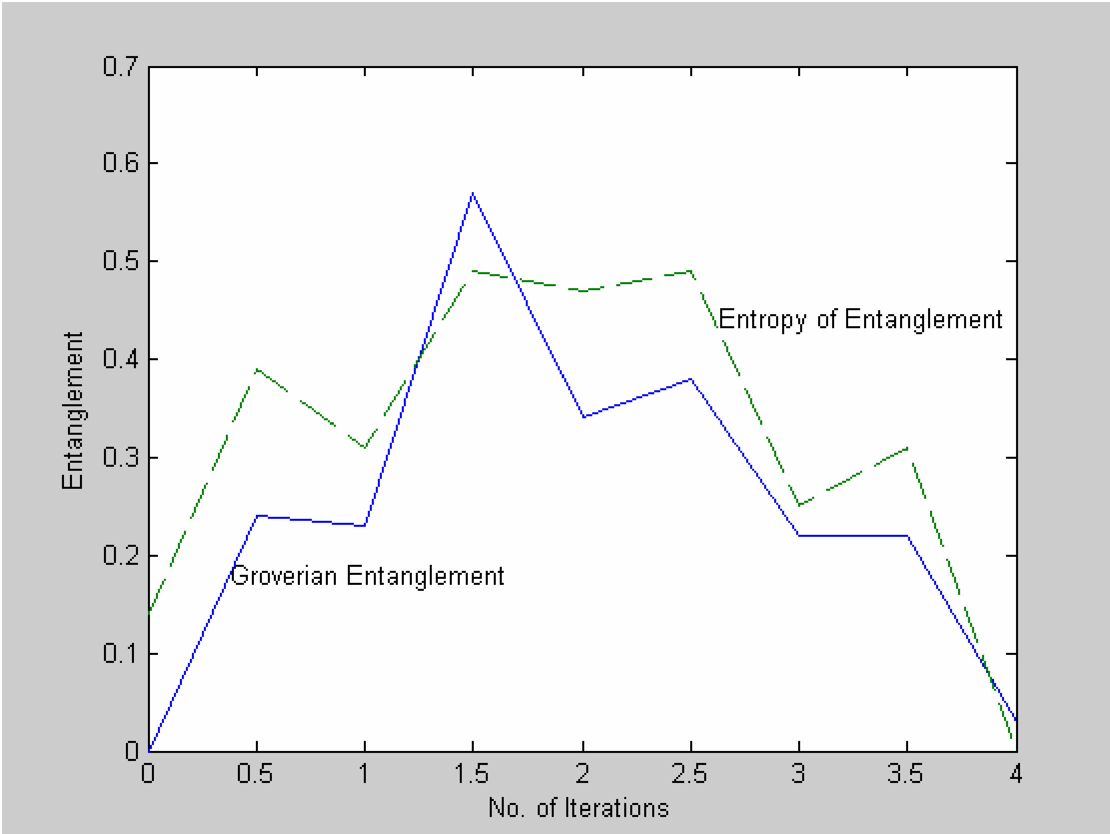

Figure2. Groverian entanglement as a function of the no. of iterations of search algorithm for five qubit system.

Discussion and conclusion

Analytical expressions for Groverian entanglement measure have been obtained for three and five -qubit states. Four qubit case has been described earlier[17]. The measure is a direct consequence of maximum success probability of Grover's search algorithm. Entanglement can be calculated for any state by considering it as the initial state of search algorithm. The fact that Grover's search algorithm performs inaccurately for entangled states has been exploited here. An analytical expression has its own benefits as the amount of entanglement can be figured out for varied choices of initial states ranging from a linear combination of two basis vectors to maximum no. of basis vectors pertaining to a given state.

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Evolution of entanglement during the iterative procedure follows a certain pattern. It rises, reaches a maximum and then decays to zero as the desired state is reached. However, the changes are not monotonic and one clearly notices the pattern of variation within each iterative step. Application of operation P_W on $|\Psi\rangle$ tends to increment the value of $G(\Psi)$ and application of P_S on the corresponding state lowers it.

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Appendix

Expression for success probability of Five-Qubit States

$$P_{max} = \frac{1}{256} [\surd(a_{00000} - a_{11000} - a_{10100} - a_{01100} - a_{10010} - a_{01010} - a_{00110} + a_{11110} - a_{10001} - a_{01001} - a_{00101} + a_{11101} - a_{00011} + a_{11011} + a_{10111} + a_{01111})^2 + (a_{10000} + a_{01000} + a_{00100} - a_{11100} + a_{00010} - a_{11010} - a_{10110} - a_{01110} + a_{00001} - a_{11001} - a_{10101} - a_{01101} - a_{10011} - a_{01011} - a_{00111} + a_{11111})^2 +$$

$$\surd(a_{00000} - a_{11000} - a_{10100} - a_{01100} - a_{10010} - a_{01010} - a_{00110} + a_{11110} + a_{10001} + a_{01001} + a_{00101} - a_{11101} + a_{00011} - a_{11011} - a_{10111} - a_{01111})^2 + (a_{10000} + a_{01000} + a_{00100} - a_{11100} + a_{00010} - a_{11010} - a_{10110} - a_{01110} - a_{00001} + a_{11001} + a_{10101} + a_{01101} + a_{10011} + a_{01011} + a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} - a_{11000} - a_{10100} - a_{01100} + a_{10010} + a_{01010} + a_{00110} - a_{11110} - a_{10001} - a_{01001} - a_{00101} + a_{11101} + a_{00011} - a_{11011} - a_{10111} - a_{01111})^2 + (a_{10000} + a_{01000} + a_{00100} - a_{11100} - a_{00010} + a_{11010} + a_{10110} + a_{01110} + a_{00001} - a_{11001} - a_{10101} - a_{01101} + a_{10011} + a_{01011} + a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} - a_{11000} - a_{10100} - a_{01100} + a_{10010} + a_{01010} + a_{00110} - a_{11110} + a_{10001} + a_{01001} + a_{00101} - a_{11101} - a_{00011} + a_{11011} + a_{10111} + a_{01111})^2 + (a_{10000} + a_{01000} + a_{00100} - a_{11100} - a_{00010} + a_{11010} + a_{10110} + a_{01110} - a_{00001} + a_{11001} + a_{10101} + a_{01101} - a_{10011} - a_{01011} - a_{00111} + a_{11111})^2 +$$

$$\surd\,(a_{00000} - a_{11000} + a_{10100} + a_{01100} - a_{10010} - a_{01010} + a_{00110} - a_{11110} - a_{10001} - a_{01001} + a_{00101} - a_{11101} - a_{00011} + a_{11011} - a_{10111} - a_{01111})^2 + (a_{10000} + a_{01000} - a_{00100} + a_{11100} + a_{00010} - a_{11010} + a_{10110} + a_{01110} + a_{00001} - a_{11001} + a_{10101} + a_{01101} - a_{10011} - a_{01011} + a_{00111} - a_{11111})^2 +$$

$$\surd\,(a_{00000} - a_{11000} + a_{10100} + a_{01100} - a_{10010} - a_{01010} + a_{00110} - a_{11110} + a_{10001} + a_{01001} - a_{00101} + a_{11101} + a_{00011} - a_{11011} + a_{10111} + a_{01111})^2 + (a_{10000} + a_{01000} - a_{00100} + a_{11100} + a_{00010} - a_{11010} + a_{10110} + a_{01110} - a_{00001} + a_{11001} - a_{10101} - a_{01101} + a_{10011} + a_{01011} - a_{00111} + a_{11111})^2 +$$

$$\surd\,(a_{00000} - a_{11000} + a_{10100} + a_{01100} + a_{10010} + a_{01010} - a_{00110} + a_{11110} - a_{10001} - a_{0100\,1} + a_{00101} - a_{11101} + a_{00011} - a_{11011} + a_{10111} + a_{01111})^2 + (a_{10000} + a_{01000} - a_{00100} + a_{11100} - a_{00010} + a_{11010} - a_{10110} - a_{01110} + a_{00001} - a_{11001} + a_{10101} + a_{01101} + a_{10011} + a_{01011} - a_{00111} + a_{11111})^2 +$$

$$\surd\,(a_{00000} - a_{11000} + a_{10100} + a_{01100} + a_{10010} + a_{01010} - a_{00110} + a_{11110} + a_{10001} + a_{01001} - a_{00101} + a_{11101} - a_{00011} + a_{11011} - a_{10111} - a_{01111})^2 + (a_{10000} + a_{01000} - a_{00100} + a_{11100} - a_{00010} + a_{11010} - a_{10110} - a_{01110} - a_{00001} + a_{11001} - a_{10101} - a_{01101} - a_{10011} - a_{01011} + a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} - a_{10100} + a_{01100} - a_{10010} + a_{01010} - a_{00110} - a_{11110} - a_{10001} + a_{01001} - a_{00101} - a_{11101} - a_{00011} - a_{11011} + a_{10111} - a_{01111})^2 + (a_{10000} - a_{01000} + a_{00100} + a_{11100} + a_{00010} + a_{11010} - a_{10110} + a_{01110} + a_{00001} + a_{11001} - a_{10101} + a_{01101} - a_{10011} + a_{01011} - a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} - a_{10100} + a_{01100} - a_{10010} + a_{01010} - a_{00110} - a_{11110} + a_{10001} - a_{01001} + a_{00101} + a_{11101} + a_{00011} + a_{11011} - a_{10111} + a_{01111})^2 + (a_{10000} - a_{01000} + a_{00100} + a_{11100} + a_{00010} + a_{11010} - a_{10110} + a_{01110} - a_{00001} - a_{11001} + a_{10101} - a_{01101} + a_{10011} - a_{01011} + a_{00111} + a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} - a_{10100} + a_{01100} + a_{10010} - a_{01010} + a_{00110} + a_{11110} - a_{10001} + a_{01001} - a_{00101} - a_{11101} + a_{00011} + a_{11011} - a_{10111} + a_{01111})^2 + (a_{10000} - a_{01000} + a_{00100} + a_{11100} - a_{00010} - a_{11010} + a_{10110} - a_{01110} + a_{00001} + a_{11001} - a_{10101} + a_{01101} + a_{10011} - a_{01011} + a_{00111} + a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} - a_{10100} + a_{01100} + a_{10010} - a_{01010} + a_{00110} + a_{11110} + a_{10001} - a_{01001} + a_{00101} + a_{11101} - a_{00011} - a_{11011} + a_{10111} - a_{01111})^2 + (a_{10000} - a_{01000} + a_{00100} + a_{11100} - a_{00010} - a_{11010} + a_{10110} - a_{01110} - a_{00001} - a_{11001} + a_{10101} - a_{01101} - a_{10011} + a_{01011} - a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} + a_{10100} - a_{01100} - a_{10010} + a_{01010} + a_{00110} + a_{11110} - a_{10001} + a_{01001} + a_{00101} + a_{11101} - a_{00011} - a_{11011} - a_{10111} + a_{01111})^2 + (a_{10000} - a_{01000} - a_{00100} - a_{11100} + a_{00010} + a_{11010} + a_{10110} - a_{01110} + a_{00001} + a_{11001} + a_{10101} - a_{01101} - a_{10011} + a_{01011} + a_{00111} + a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} + a_{10100} - a_{01100} - a_{10010} + a_{01010} + a_{00110} + a_{11110} + a_{10001} - a_{01001} - a_{00101} - a_{11101} + a_{00011} + a_{11011} + a_{10111} - a_{01111})^2 + (a_{10000} - a_{01000} - a_{00100} - a_{11100} + a_{00010} + a_{11010} + a_{10110} - a_{01110} - a_{00001} - a_{11001} - a_{10101} + a_{01101} + a_{10011} - a_{01011} - a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} + a_{10100} - a_{01100} + a_{10010} - a_{01010} - a_{00110} - a_{11110} + a_{10001} + a_{01001} + a_{00101} + a_{11101} + a_{00011} + a_{11011} + a_{10111} - a_{01111})^2 + (a_{10000} - a_{01000} - a_{00100} - a_{11100} - a_{00010} - a_{11010} - a_{10110} + a_{01110} + a_{00001} + a_{11001} + a_{10101} - a_{01101} + a_{10011} - a_{01011} - a_{00111} - a_{11111})^2 +$$

$$\surd(a_{00000} + a_{11000} + a_{10100} - a_{01100} + a_{10010} - a_{01010} - a_{00110} - a_{11110} - a_{10001} - a_{01001} - a_{00101} - a_{11101} - a_{00011} - a_{11011} - a_{10111} + a_{01111})^2 + (a_{10000} - a_{01000} - a_{00100} - a_{11100} - a_{00010} - a_{11010} - a_{10110} + a_{01110} - a_{00001} - a_{11001} - a_{10101} + a_{01101} - a_{10011} + a_{01011} + a_{00111} + a_{11111})^2]^2$$